A Comprehensive Guide to Compton Scattering

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Introduction and some background information

Arthur Holly Compton made a striking realization in 1923. He discovered that when a high frequency photon (particle of light) interacts with a charged particle, like an electron, it results in a change of the photon's wavelength. The name of this phenomenon is Compton scattering, when the photon's energy decreases (it's wavelength increases) it is usually referred to as Compton effect. The understanding of this phenomenon is indispensable to radiobiology as high energy gamma rays and X-rays mostly interact with atoms of our bodies through Compton scattering.

Prerequisite knowledge

To understand the Compton effect one must understand the concept of Special Relativity and be familiar with the basics of particle physics. In 1905 or the "annus mirabilis" (miracle year) Albert Einstein published four groundbreaking papers, one of them was titled: "On the Electrodynamics of moving bodies". This later became known as "Special Relativity". Newton's laws of motion are so accurate at everyday scales that nobody thought that they might be wrong, but that is what Einstein showed.

Special Relativity

Some definitions that will be necessary to understand this section:

- **Frame of reference**: A coordinate system whose origin, orientation and scale are specified by a set of reference points whose position is both mathematically and physically defined.
- **Vector**: In physics vectors are used to represent quantities that have both a magnitude and a direction, such as velocities or forces.
- **Causality**: Causality is influence by which one event, process (a cause) contributes to the production of an other event, process (an effect).
- **Observer**: An observer is a frame of reference from which a set of objects or events are being measured.

So, now that we are familiar with the necessary definitions we can continue with Einstein's theory. Galileo Galilei introduced his principle of relativity even before Newton. Galilean relativity states that the laws of motion are the same in all inertial frames of reference.[1] Newton, who built on these ideas, introduced the concept of absolute space (there exits a reference frame that is always similar and immovable) and absolute time (time flows at the same rate for every observer regardless to anything external). Now, let us imagine a reference frame which is at rest and another one which is moving at a constant speed relative to it (primed) along the x-axis (one-dimensional motion). Galileo would link the two reference frames by the following equations:

$$x' = x - vt, \qquad y' = y, \qquad z' = z, \qquad t' = t \tag{1}$$

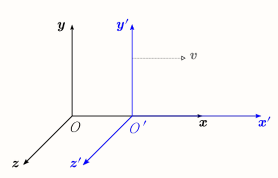


Figure 1: The two frames in relative motion

Einstein based on insights from James Clerk Maxwell and Hendrik Lorentz introduced two principles that form the base of Special Relativity:

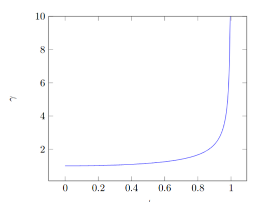
1. The laws of physics are invariant (identical) in all inertial frames of reference.
2. The speed of light in vacuum is the same for all observers, regardless of the motion of light source or observer.[2]

The constancy of the speed of light modifies the way we can connect the two reference frames the following way:

$$x' = \gamma(x - vt), \qquad y' = y, \qquad z' = z, \qquad t' = \gamma\left(t - \frac{v}{c^2}x\right) \tag{2}$$

$$\text{where} \qquad \boxed{\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}}$$

These set of equations is called the Lorentz transformation. As you can see, due to Einstein's second principle the equations become a little more complicated. "c" denotes the speed of light in a vacuum and "γ" is called *Lorentz gamma factor*. Notice, that $\gamma(0) = 1$ and $\gamma(c) = \infty$, so $\gamma \geq 1$. If we input a larger value than c we get that:$\gamma \notin \mathbb{R}$, which is not physical.



Speed (units of c)	Value of γ
0.100	1.005
0.400	1.091
0.850	1.898
0.950	3.203
0.999	22.366

Figure 2: The values of γ for different velocities

There is one more equation from Special relativity that we must look at:

$$E_0 = Mc^2 \tag{3}$$

$$\text{If there is motion} \qquad E_{total} = \gamma Mc^2 \tag{4}$$

This is probably the most famous equation in all of physics, Einstein's mass-energy equivalence. Looking at the formula we can tell that even a small amount of mass corresponds to great quantities of energy. If the particle with mass M is moving relative to our reference frame we have to use equation (4), which shows us the total energy of the given particle. Because of this the formula for kinetic energy gets modified as well: $\boxed{E_{kin} = (\gamma - 1)Mc^2}$. Based on our insights we can say that an object with mass cannot reach the speed of light as it would need and infinite amount of energy. We can also discover that at low speeds Lorentz transformation turns into Galilean transformation.

Consequences of Lorentz transformation

Einstein's theory comes with numerous surprises. In this section we will explore 3 of them.

- **Causality**: Consider two events $A(x_A; t_A)$ and $B(x_B; t_B)$ in spacetime[3], so we have one spacial- and one temporal dimension. If A is the cause for B, then we have to show that $t_A \leq t_B$ in all coordinate systems. From the Lorentz transformation we know:

$$\Delta t' = t'_B - t'_A = \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right) \geq 0$$
$$\Rightarrow \quad 1 - \frac{v}{c^2}\frac{\Delta x}{\Delta t} \geq 0 \qquad \text{Where} \quad \frac{\Delta x}{\Delta t} = v$$
$$\Rightarrow \quad 1 - \frac{v^2}{c^2} \geq 0$$
$$\Rightarrow \quad |v| \leq c$$

  Therefore we can conclude that causality is preserved in Lorentz Transformation.

- **Lorentz Contraction**: Let us imagine a rod with length $\Delta x = L_0$ in frame K which moves at speed v in a reference frame K′. We will denote the measured length in K′ by $\Delta x' = L$. We will measure the endpoints of the rods simultaneously, so $\Delta t = 0$. In this case the Lorentz transformation is:

$$\Delta x' = \gamma(\Delta x - v\Delta t)$$
$$\Rightarrow \quad L_0 = \gamma L$$
$$\Rightarrow \quad \boxed{L = \frac{L_0}{\gamma}} < L_0$$

  So the moving stick will appear shorter in K′, however, this conclusion is valid for all objects.

- **Time dilation**: Consider two events in the same place in frame K′ which moves with speed v in frame K. The events are: (x', t'_1) and (x', t'_2). Because the two events are in the same location: $\Delta x'$=0 and $\Delta t' = t'_2 - t'_1$. So the Lorentz transformation gives:

$$\Delta t = \gamma\left(\Delta t' - \frac{v}{c^2}\Delta x'\right) = \boxed{\gamma \Delta t'} > \Delta t'$$

  So we can conclude that a moving "clock" will run slower when viewed from K, however, this is true for any physical phenomena.

Four-vectors

In physics spacetime is a mathematical model that combines the three dimensions of space and one dimension of time. In this section we are going to look at vectors in spacetime, called Four-vectors, as they have four components instead of three.

Now, can we find an algebraic expression for a quantity that is invariant under Lorentz transformation? The answer is yes and this quantity is called *spacetime interval*:

$$c^2t^2 - x^2 - y^2 - z^2 \tag{5}$$

Invariant means that regardless of the reference frame this quantity will remain the same. This is also called *Lorentz invariance*. There are two other concepts we must not ignore. The *worldline* of an object is the path that an object traces in 4-dimensional spacetime. The other one is *proper time*. In relativity proper time along a world line is defined as the time as measured by a clock following that line, it is independent of coordinates and the same for all observers.

With this in mind consider two events (x_1, t_1) and (x_2, t_2) and define $\Delta x = x_2 - x_1$ and $\Delta t = t_2 - t_1$. The difference in proper time $\Delta\tau$ is given by:

$$c^2\Delta\tau^2 = c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 = \text{invariant} \tag{6}$$

The value of $\Delta\tau^2$ is important:

- $\Delta\tau^2 > 0 \Rightarrow$ the events are timelike, and the events can be causally connected via particles with mass.
- $\Delta\tau^2 = 0 \Rightarrow$ the events are lightlike, and the events can be causally connected via light.
- $\Delta\tau^2 < 0 \Rightarrow$ the events are spacelike, and the events can not be causally connected.

The way we carry out calculations with Four-vectors can be easily understood if we first get ourselves familiar with calculating with three-dimensional vectors.

Let us consider two vectors in three dimensions of space: $\vec{v} = (v_1, v_2, v_3)$ and $\vec{u} = (u_1, u_2, u_3)$. In physics we denote vectors with an arrow and inside the brackets we have the components of the vector (in this case three as we are in 3D space). One operation that we can carry out on our vectors is called a *dot product*. The dot product of our vectors:

$$\vec{v} \cdot \vec{u} = v_1u_1 + v_2u_2 + v_3u_3 \tag{7}$$

The length of the vector $\vec{r} = (x, y, z)$ is given by:

$$r = |\vec{r}| = \sqrt{\vec{r} \cdot \vec{r}} = \sqrt{x^2 + y^2 + z^2} \tag{8}$$

With this in mind we can turn our attention to four-vectors, which are four dimensional (three dimensions of space and one of time). We have the following four-vectors: A=(ct_A, x_A, y_A, z_A)=$(ct_A, \vec{r_A})$ and B=(ct_B, x_B, y_B, z_B)=$(ct_B, \vec{r_B})$. For four-vectors our notation changes to capital to distinguish them from three-dimensional ones. The dot product in (3+1)D is given by:

$$A \cdot B = c^2t_At_B - \vec{r_A} \cdot \vec{r_B} = c^2t_At_B - x_Ax_B - y_Ay_B - z_Az_B \tag{9}$$

The four-vector A is timelike if $A \cdot A > 0$, lightlike if $A \cdot A = 0$ and spacelike if $A \cdot A < 0$.

In newtonian physics **momentum** (denoted with a "p") is a vector in 3D space. In special relitivity we can write **four-momentum**[4] in spacetime the following way:

$$p = \left(\frac{E}{c}, p_x, p_y, p_z\right) = \left(\frac{E}{c}, \vec{p}\right) \tag{10}$$

Defining momentum in special relativity is necessary to be able to calculate particle interactions like the Compton effect. Similarly to the spacetime interval we can find an invariant quantity in Lorentz transformation for four-momentum. This invariant quantity is "four-momentum squared":

$$p^2 = p \cdot p = \frac{E^2}{c^2} - p_x^2 - p_y^2 - p_z^2 \tag{11}$$

Now, let us use this fact to arrive at a formula which we will be relying on when contemplating particle interactions. Consider a frame, where a particle with mass M is stationary. There $p_0 = (\frac{E}{c}, \vec{0}) = (Mc, \vec{0})$, and $p_0^2 = p_o \cdot p_0 = M^2c^2$. Since p^2 is invariant we have:

$$p^2 = p_0^2$$
$$\frac{E^2}{c^2} - |\vec{p}|^2 = M^2c^2 \qquad / \cdot c^2$$
$$E^2 = (Mc^2)^2 + (\vec{p}c)^2 \tag{12}$$

This is the *total energy* of a particle with mass M in motion with momentum $\vec{p}$. Notice, if there is no motion ($\vec{p} = 0$) we get the famous $E = mc^2$ and for massless particles the equation gives $E = \vec{p}c$. For all real particles the four-momentum squared is $p^2 = M^2$, also for massless particles. If this condition is satisfied, we say that the particle is *on mass shell* or *on-shell*. However, when we take quantum effects into account, we find particles that violate this condition. These particles are therefore *off-shell*[5].

Relativistic kinematics

In this section we will look at *relativistic kinematics*. Relativistic means that it is a framework compatible with special relativity. Kinematics is an area of physics that describes the motion of points and objects (bodies). So with this "tool" we will be able to provide a non-quantum mechanical description of a system of particles, in cases where the velocities of moving objects are comparable to the speed of light "c". The energy of a photon was first described by Max Planck in 1900[6]. The formula he gave is this:

$$E = hf = h\frac{c}{\lambda} \tag{13}$$

f is the frequency and λ is the wavelength of the photon. Since photons have no mass, from the total energy formula we read that the energy of the photon is $E = pc$, where p is the momentum of the photon. If we substitute this into Planck's formula, we get:

$$pc = h\frac{c}{\lambda}$$
$$\boxed{p = \frac{h}{\lambda}} \tag{14}$$

so the definition of momentum can be extended to massless particles as well. Similarly all massive particles can be connected to a certain wavelength by solving the equation above with respect to λ. Just like photons, every particle exhibits wave-like properties with a wavelength $\lambda = h/mv$, if $v \ll c$. This startling concept was first proposed in 1924 by Louis de Broglie, further investigations of this property lead to the Schrödinger equation and the birth of *quantum mechanics*.

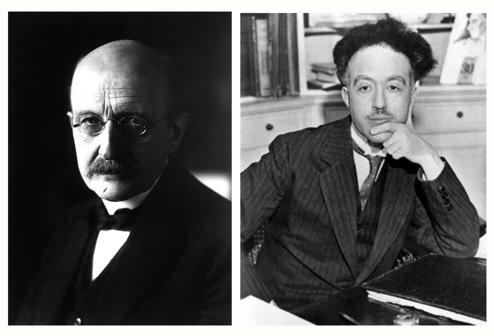


Figure 3: Max Planck (left) and Louis de Broglie (right)

Since the Compton effect involves scattering of particles, it is compulsory for us to go through the basics of collisions. If the particles involved in the scattering process are the same in the initial state and in the final state, the scattering is *elastic*. Otherwise we have one or more particles created or destroyed in the process, in which case the scattering is *inelastic*. However, the total energy and the total momentum of the system will always be conserved in particle interactions.

Compton scattering

Compton scattering is a phenomenon where electromagnetic radiation loses energy when it scatters from a charged particle. The incoming radiation, in the form of a photon, changes it's wavelength (for a different energy level the wavelength is different, which can be seen from Planck's formula). We are going to study the following interaction:

$$e^- + \gamma \rightarrow e^- + \gamma$$

Compared to pion creation, no new particle is created. Instead, the incoming photon collides with the electron, as a result the wavelength of the photon will change. This change of wavelength should not happen in classical physics. Let us calculate the change of photon wavelength upon collision, and how it relates to the scattering angle θ. Here θ is the angle between the initial and final photon momentum vectors.

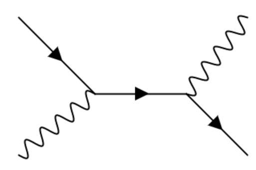


Figure 4: The Feynman diagram of Compton scattering. Here time flows from left to right, the straight line marks the electron, and the "wavy" one represents the photon. The electron absorbs a photon and emits it soon after at longer wavelengths.

Derivation of Compton's formula

Let us denote the four-momentum of the photon before the scattering as k, and after the scattering as k′. Will employ a similar notation for the electron's four-momenta: p before, and p′ after scattering. These vectors have the following value:

$$k = (|\vec{k}|, \vec{k}), \qquad k' = (|\vec{k'}|, \vec{k'}), \qquad p = \left(\frac{E_e}{c}, \vec{0}\right), \qquad p' = \left(\frac{E'_e}{c}, \vec{p'}\right).$$

It is important to note that we treat the electron as stationary before the scattering. Now, we will also consider the energies of our particles, as energy conservation is necessary in this interaction. We will use the following notation for the energies, wavelength and frequencies:

$$E_\gamma = hf = \frac{hc}{\lambda}, \qquad E'_\gamma = hf' = \frac{hc}{\lambda'}, \qquad E_e = mc^2, \qquad E'_e = \sqrt{(mc^2)^2 + (|\vec{p'}|c)^2}.$$

Unlike in classical physics, a photon has nonzero momentum (from the total energy formula). Now, that we have our four-momenta and energies, we can make use of energy conservation (the total energy before and after the scattering has to be equal):

$$E_\gamma + E_e = E'_\gamma + E'_e$$
$$hf + mc^2 = hf' + \sqrt{p'^2c^2 + m^2c^4}$$
$$hf - hf' + mc^2 = \sqrt{p'^2c^2 + m^2c^4} \qquad /()^2$$
$$(h(f - f') + mc^2)^2 = p'^2c^2 + m^2c^4$$
$$\Rightarrow p'^2c^2 = (h(f - f') + mc^2)^2 - m^2c^4$$

This result will be enormously useful. We can now utilize momentum-conservation and use the relation above. The change in the photon's momentum must equal the momentum gained by the electron in the scattering:

$$\vec{k} - \vec{k'} = \vec{p'} \qquad /()^2$$
$$|\vec{k}|^2 + |\vec{k'}|^2 - 2\vec{k} \cdot \vec{k'} = |\vec{p'}|^2$$
$$|\vec{k}|^2 + |\vec{k'}|^2 - 2|\vec{k}||\vec{k'}|\cos\theta = |\vec{p'}|^2$$
$$\left(\frac{h}{\lambda}\right)^2 + \left(\frac{h}{\lambda'}\right)^2 - 2\left(\frac{h}{\lambda}\right)\left(\frac{h}{\lambda'}\right)\cos\theta = |\vec{p'}|^2 \qquad / \cdot c^2$$
$$\left(\frac{hc}{\lambda}\right)^2 + \left(\frac{hc}{\lambda'}\right)^2 - \frac{2h^2c^2}{\lambda\lambda'}\cos\theta = (h(f - f') + mc^2)^2 - m^2c^4 \Longleftarrow$$
$$\left(\frac{hc}{\lambda}\right)^2 + \left(\frac{hc}{\lambda'}\right)^2 - \frac{2h^2c^2}{\lambda\lambda'}\cos\theta = \left(\frac{hc}{\lambda}\right)^2 + \left(\frac{hc}{\lambda'}\right)^2 + m^2c^4 + 2\frac{hc}{\lambda}mc^2 - 2\frac{hc}{\lambda'}mc^2 - 2\frac{h^2c^2}{\lambda\lambda'} - m^2c^4$$
$$-\frac{2h^2c^2}{\lambda\lambda'}\cos\theta = 2\frac{hc}{\lambda}mc^2 - 2\frac{hc}{\lambda'}mc^2 - 2\frac{h^2c^2}{\lambda\lambda'}$$
$$\frac{h^2}{\lambda\lambda'}(1 - \cos\theta) = \frac{hc}{\lambda}m - \frac{hc}{\lambda'}m \qquad / : (hcm)$$
$$\frac{h}{cm\lambda\lambda'}(1 - \cos\theta) = \frac{1}{\lambda} - \frac{1}{\lambda'}$$
$$\frac{h}{cm\lambda\lambda'}(1 - \cos\theta) = \frac{\lambda' - \lambda}{\lambda\lambda'} \qquad / \cdot \lambda\lambda'$$
$$\frac{h}{cm}(1 - \cos\theta) = \lambda' - \lambda$$
$$\Rightarrow \lambda_C(1 - \cos\theta) = \Delta\lambda$$

where $\lambda_C = h/cm$ is the *Compton wavelength*. We have successfully arrived at the relation, that earned Arthur Holly Compton a Nobel Prize in 1927!

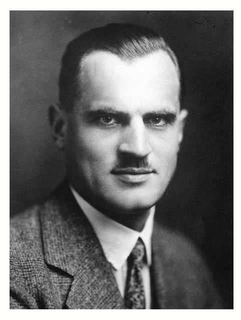


Figure 5: Arthur Holly Compton in 1927

The importance of Compton scattering

This finding is significant, because it demonstrates that light cannot be thought of as a purely wave phenomenon. Classical theories could not explain the shift in wavelength at low intensity. To explain our results we must conclude, that light behaves in these interactions as a particle. Photons can interact with matter through various processes, one of them being Compton scattering. At energies of a few eV to a few Kev, a photon can be absorbed and it's energy can eject an electron, this is the famous *photoelectric effect*, described by Einstein. In the high-energy regions (from 1.022 Mev), we first come across *pair production*, and at even higher energies we discover *photodisintegration*[7]. In the energy regions between the photoelectric effect and pair production, Compton scattering is the most important interaction.

Used sources

The following Wikipedia pages were read on 11th of February:

- https://en.wikipedia.org/wiki/Compton_scattering
- https://en.wikipedia.org/wiki/Frame_of_reference
- https://en.wikipedia.org/wiki/Vector_(mathematics_and_physics)
- https://en.wikipedia.org/wiki/Observer_(special_relativity)
- https://en.wikipedia.org/wiki/Spacetime
- https://en.wikipedia.org/wiki/Special_relativity
- http://astro.vaporia.com/start/relativisticinvariance.html
- https://en.wikipedia.org/wiki/World_line
- https://en.wikipedia.org/wiki/Proper_time
- https://en.wikipedia.org/wiki/Relativistic_mechanics
- https://en.wikipedia.org/wiki/Electronvolt
- https://en.wikipedia.org/wiki/Center-of-momentum_frame
- Figure 1:https://en.wikipedia.org/wiki/File:Frames_of_reference_in_relative_motion.svg
- Figure 2 was generated on the following website: https://feynman.aivazis.com
- Image of Arthur Compton: https://commons.wikimedia.org/wiki/File:Arthur_Compton_1927.jpg
- Planck https://commons.wikimedia.org/wiki/File:Max_Planck_(1858-1947).jpg
- de Broglie https://commons.wikimedia.org/wiki/File:Broglie_Big.jpg

[1] Inertial means, that the reference frame is under no acceleration
[2] The speed of light in a vacuum, often denoted as "c", is 299792458 $\frac{m}{s}$
[3] In physics spacetime is a mathematical model that combines the three dimensions of space and one dimension of time
[4] We will still denote four-momentum with a "p"
[5] These particles are called *virtual particles*, which we will not explore here.
[6] The year 1900 is often called the birth of quantum mechanics. Planck's equation tells us that light is quantized, it comes in energy "packets"
[7] In pair production an electron and positron pair may be created. At higher energies, the bombarded nucleus can eject a nucleon or alpha particle, this is photodisintegration.